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Phase Transformations in Quaternionic Quantum Field Theory

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Phase transformations of a scalar quaternionic quantum field are examined as unitarily implemented symmetries. Under very general quantization conditions it is shown, in both global and local cases, that the only sensible phase invariance that has been suggested is $\phi \to p\phi p^{-1}$, where p is a quaternion and ϕ a quaternionic scalar field.

1. INTRODUCTION

Phase transformations, that is, transforming fields by multiplying the fields by the scalars of the theory, are basic to present-day quantum theories. So, when examining quaternionic quantum theories many authors introduce some sort of quaternionic phase invariance (Horwitz and Biedenharn, 1984; Rembielinski, 1981; Adler, 1985; Morita, 1982; Kaneno, 1960; Finkelstein *et al.*, 1963). In fact, a number set out with the idea of introducing a new phase invariance in mind (Morita, 1982; Kaneno, 1960; Finkelstein *et al.*, 1963). Two types of phase transformations have been suggested for a quaternionic quantum theory. First, there are those sets of transformations that contain as a subset the set of transformations $\phi \to p\phi$, where p is any unit quaternion and ϕ is a field (Horwitz and Biedeharn, 1984; Adler, 1985; Morita, 1982; Kaneno, 1960). Second, there is the set of transformations $\phi \to p\phi p^{-1}$, where p is any unit quaternion (Finkelstein *et al.*, 1963).

In this paper we investigate these proposed phase transformations specifically as symmetries of a quantum field theory. Quantum field theories characteristically feature field operators $\phi(x)$ (acting on a Hilbert space) defined for each element x of spacetime S. This Hilbert space will of course

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be quaternionic.[2] These field operators are in general characterized by commutation relations. Depending upon how general one wishes to be, these commutation relations may be taken as either the canonical commutation relations (CCR)

$$[\phi^+(x), \phi(y)]_\pm = \delta(x-y)$$

$$[\phi(x), \phi(y)]_\pm = 0$$

for all x and $y \in S$ with $x_0 = y_0$, or the spacelike-separated commutation relations

$$[\phi^+(x), \phi(y)]_\pm = 0$$

$$[\phi(x), \phi(y)]_\pm = 0$$

for all x and y spacelike-separated elements of S (Streater and Wightman, 1964).

We will use the following conventions: Unless otherwise specified, Roman letters will not be summed and Greek letters will be summed. Unless otherwise specified, quaternions will be assumed to be of unit length; in which case $p^* = p^{-1}$.

In discussing symmetries we are principally concerned with ray transformations of the Hilbert space that leave the inner product alone. Fortunately, Emch (1963) has shown that any such ray transformation can be induced by a unitary operator unique up to a sign. So we will consider symmetries as unitarily implemented.

2. MULTIPLICATION ON THE LEFT BY A QUATERNION

2.1. Global Symmetry

First we will consider the set of transformations $\{\phi_i \to p\phi_i : p \in Q, |p| = 1\}$ as a set of unitarily implemented symmetry transformations. We will assume that the fields satisfy the discrete form of the CCRs $[\phi_i^+, \phi_j]_\pm = \delta_{ij}$ and $[\phi_i, \phi_j]_\pm = 0$ for all i, j elements of some discrete set I. Note that i is taking the place of x as the parameter on which the field depends. Assuming for all i that the operator $\phi_i^+\phi_i$ has an eigenvector, then, just as with complex theories, one may build up a Fock space of eigenvectors of $\phi_i^+\phi_i$. In the following the expression "eigenvector of all the $\phi_i^+\phi_i$" will be replaced by E.A.$\phi^+\phi$.

[2]For an excellent introduction to quaternionic Hilbert spaces and operators on them see Horwitz and Biedenharn (1984). It is important to note that we will be using the same definition of scalar multiplication as Horwitz and Biedenharn (1984) that is, with the quaternion on the right.

We will assume that no two nonparallel vectors that are both E.A.$\phi^+\phi$ have the same eigenvalues for all the operators $\phi_i^+\phi_i$. This means that there is only one state for each set $\{\lambda_i : i \in I\}$ of eigenvalues of $\phi_i^+\phi_i$, just as is assumed in complex theories.

Since $\phi_i \to p\phi_i$ is unitarily implemented for all quaternions p, then $[(p\phi_i)^+, p\phi_j]_\pm = \delta_{ij}$ for all p. A little manipulation shows for both the plus and the minus cases that $p\phi_i^+\phi_j = \phi_i^+\phi_j p$ for all p and for all i and j. This implies that $\phi_i^+\phi_j$ and similarly $\phi_j\phi_i^+$ are real operators for all i and j. Now consider V an E.A.$\phi^+\phi$. Because $\phi_i^+\phi_i$ is Hermitian for all i, the eigenvalues are all real. Writing $V = v_\alpha e_\alpha$, where the v_α are real vectors, we have, for $\phi_i^+\phi_i V = V\lambda_i$,

$$\phi_i^+\phi_i v_\alpha e_\alpha = \phi_i^+\phi_i V = V\lambda_i = v_\alpha e_\alpha \lambda_i = v_\alpha \lambda_i e_\alpha$$

Thus, as real operators send real vectors to real vectors, we have $\phi_i^+\phi_i v_\alpha = v_\alpha\lambda_i$. However, because we have assumed V, being the eigenvector associated with $\{\lambda_i : i \in I\}$ of eigenvalues, to be unique up to scalar multiplication, then we have that $V = vq'$ for some quaternion q' and some real vector v. Now let v, v', and v'' always represent real E.A.$\phi^+\phi$.

If v is a real E.A.$\phi^+\phi$ and if j is such that $\phi_j^+ v \neq 0$, then $\phi_j^+ v$ is an E.A.$\phi^+\phi$. So $\phi_j^+ v = v'q$ for some quaternion q and some real vector v' as above.

Let U_p be the unitary operator that induces the transformation $\phi_j \to p\phi_j = U_p\phi_j U_p^+$ for all j. Then, as

$$\begin{aligned} U_p\phi_j^+\phi_j U_p^+ &= U_p\phi_j^+ U_p^+ U_p\phi_j U_p^+ \\ &= (U_p\phi_j U_p^+)^+ U_p\phi_j U_p^+ = (p\phi_j)^+ p\phi_j \\ &= \phi_j^+ p^* p\phi_j = \phi_j^+\phi_j \qquad \text{for all } j \end{aligned}$$

U_p commutes with $\phi_j^+\phi_j$ for all j. So U_p must leave the eigenspaces of $\phi_j^+\phi_j$ invariant for all j. Therefore, if V is an E.A.$\phi^+\phi$, then $U_p V = VS_p^V$ for some unit quaternion S_p^V because V is unique as above.

Take $j \in I$, suppose that v is a real E.A.$\phi^+\phi$ such that $\phi_j^+ v \neq 0$, and let $\phi_j^+ v = v'q$ for some real E.A.$\phi^+\phi$ v' and quaternion q. So

$$U_p\phi_j^+ U_p^+ v = U_p\phi_j^+ v(S_p^v)^* = U_p v' q(S_p^v)^* = v' S_p^{v'} q(S_p^v)^*$$

and

$$U_p\phi_j^+ U_p^+ v = \phi_j^+ p^* v = \phi_j^+ v p^* = v' q p^*$$

Therefore

$$S_p^{v'} = qp^* S_p^v q^* \tag{1}$$

Now we are in a position to show that we can always find quaternions p and r such that $U_pU_r\phi_iU_r^+U_p^+$ is not of the form $w\phi_i$ for any quaternion w.

To show this, we will assume the opposite, that is, that for all quaternions p and r there exists a quaternion w such that $U_pU_r\phi_iU_r^+U_p^+ = w\phi_i$ for all i. As a result, we have

$$w\phi_i = U_pU_r\phi_iU_r^+U_p^+ = U_pr\phi_iU_p^+ = U_prU_p^+U_p\phi_iU_p^+ = U_prU_p^+p\phi_i \qquad (2)$$

for all i and all p and r. Now, for any $m \neq i$ we can find v a real E.A.$\phi^+\phi$ such that there exist vectors V_1 and V_2 E.A.$\phi^+\phi$ for which $v = \phi_iV_1$ and $\phi_m^+v = \phi_iV_2$. Putting V_1 into equation (2), we find that

$$S_p^v r(S_p^v)^* pv = U_prU_p^+pv = wv$$

Putting V_2 into equation (2) and remembering $\phi_m^+v = v'q$ for some quaternion q and for v' some real E.A.$\phi^+\phi$, we have

$$S_p^{v'} r(S_p^{v'})^* pv'q = U_prU_p^+pv'q = wv'q$$

So $S_p^v r(S_p^v)^*p = S_p^{v'} r(S_p^{v'})^*p$ for all p and r. Therefore, $[r, (S_p^v)^*S_p^{v'}] = 0$ for all p and all r, which shows from equation (1) that $S_p^v = S_p^{v'} = qp^*S_p^vq^*$ for all p. As we can set p equal to q, q must equal 1. Therefore, for arbitrary p, $S_p^v = p^*S_p^v$, which is a contradiction, as $S_p^v \neq 0$.

The first thing to note about this result is that it means that the unitary operators U_p cannot represent the group of transformations, because they are not closed under multiplication. This makes the theory very different from complex theories, where this sort of representation of the group of transformations is assumed. However, this is not to say that the theory is necessarily untenable.

The second thing to note is that for the transformed fields $r\phi_i$, the association between their transformation properties and conserved charges is confused because $U_pr\phi U_p^+$ cannot always be written as a simple function of $r\phi$ for all p. Though it is quite clear that there are quaternions r such that $r\phi_i$ does not transform as ϕ_i, it is not clear which $r\phi$ and $r'\phi$ transform in the same way. It seems likely, however, that there are infinitely many $r\phi$ transforming differently for each r. If this were the case, the theory would then have infinitely many different particles associated with it, which would be unacceptable. On the other hand, the theory would seem to be rendered unacceptable from another problem: it appears that only the identity commutes with U_p for all p. So then the only possible time translation operator is trivial.

2.2. Gauge Symmetry

Not only can the transformations $\phi \to p\phi$ be rejected as a global symmetry of a field theory satisfying CCRs, but also $\phi \to p\phi$ can be rejected as a gauge symmetry of a field theory satisfying the less restrictive and better motivated spacelike-separated commutation relations. As a gauge symmetry we assert that each element of the set of transformations $\{\phi(x) \to p(x)\phi(x)$: $p(x)$ is a unit quaternion for all $x \in S\}$ is implemented by a unitary symmetry operator. Then, because of the unitary impementation of the transformations, the transformed fields will always satisfy the relation $[(p(x)\phi(x))^+, p(y)\phi(y)]_\pm = 0$ for all spacelike-separated x and $y \in S$. Now, for the moment fix x and y; then, by choosing in turn the transformation such that $p(x) = 1$ and then $p(y) = 1$ and using $[\phi^+(x), \phi(y)]_\pm = 0$, we arrive at

$$[\phi^+(x), p(y)]_-\phi(y) = 0 = \phi^+(x)[p(x)^*, \phi(y)]_-$$

for all values of $p(y)$ and $p(x)$. By successively choosing $p(y) = e_1$, e_2, and e_3 and $p(x) = e_1$, e_2, and e_3 it can be shown by tedious expansions that $\phi^+(x)_\alpha\phi(y)_\beta = 0$, α, $\beta = 0, 1, 2, 3$, except for $0 = \alpha = \beta$. Here we are using $\phi(y) = \phi(y)_\beta e_\beta$ and similarly for $\phi^+(x)$. That $\phi^+(x)_0\phi(y)_0 = 0$ can be shown by looking back at the original condition. So, as x and y were any spacelike-separated elements of spacetime, then $\phi^+(x)\phi(y) = 0$ for all such x and y.

To show that this is unacceptable for a physical theory, we will make the following added assumptions:

1. At least one spacelike translation $\phi(x) \to \phi(x+a)$ is unitarily implemented, i.e., $\phi(x+a) = U\phi(x)U^+$ for all $x \in S$ for some spacelike a and for some unitary U.
2. There exists a vector V_0 called the vacuum such that $UV_0 = V_0$ and there exists a $z \in S$ such that $\phi^+(z)V_0 \neq 0$.

From these assumptions we have that

$$(V_0, \phi(z)\phi^+(z)V_0) = (\phi^+(z)V_0, \phi^+(z)V_0) = \|\phi^+(z)V_0\|^2 > 0$$

Therefore

$$\phi(z)\phi^+(z)V_0 = V_0\frac{\|\phi^+(z)V_0\|^2}{\|V_0\|^2} + V = V_0C + V$$

for some V, where $(V, V_0) = 0$ and $C = \|\phi^+(z)V\|^2/\|V_0\|^2$. Now for any integer $n \neq 0$ we have

$$\begin{aligned}(U^n(V_0C+V), V_0C+V) &= (U^n\phi(z)\phi^+(z)V_0, \phi(z)\phi^+(z)V_0)\\ &= (\phi(z+na)\phi^+(z+na)V_0, \phi(z)\phi^+(z)V_0)\\ &= (\phi^+(z+na)V_0, \phi^+(z+na)\phi(z)\phi^+(z)V_0)\\ &= 0\end{aligned}$$

as $\phi^+(z+na)\phi(z)=0$, since $z+na$ is spacelike-separated from z. It follows that

$$
\begin{aligned}
0 &= (U^n V_0 C + U^n V,\ V_0 C + V) \\
&= (V_0 C,\ V_0 C) + (V_0 C,\ V) + (U^n V,\ V_0 C) + (U^n V,\ V)
\end{aligned}
$$

Then, as $(U^n V,\ V_0 C) = (V,\ U^{n+} V_0 C) = (V,\ V_0 C) = 0$, we have

$$
(U^n V,\ V) = -\|V_0\|^2 C^2 = -\frac{\|\phi^+(z) V_0\|^4}{\|V_0\|^2} < 0 \qquad \text{for all} \quad n \neq 0
$$

This is shown in the Appendix to be a contradiction for any unitary operator. The above contradiction may be arrived at by a practically unchanged argument in the more general situation where we consider the fields as dependent not on spacetime points, but on functions, as is done in axiomatic field theory (Streater and Wightman 1964). So it is not a result of the use of spacetime points.

3. AUTOMORPHISMS AS A SYMMETRY

This leaves us to examine the set of transformations $\{\phi \to p\phi p^*: p$ a unit quaternion$\}$ as set of unitarily implemented symmetry transformations. This set is isomorphic to the group of automorphisms of the quaternions. Now, multiplication of vectors by unit quaternions on the left is a unitary transformation (Horwitz and Biedenharn, 1984). So, p is the unitary transformation that implements $\phi(x) \to p\phi(x)p^*$. If $p\phi(x)p^* = p'\phi(x)p'^*$, then $[p'^*p, \phi(x)] = 0$, so, provided $\phi(x)$ does not have a fixed imaginary direction independent of x, it must be that $p = p'$ and so p is unique. Then the unitary operators represent the group of transformations faithfully as $U_p U_q = pq = U_{pq}$. This also means that $p\phi p^*$ transforms in the same way as ϕ. For a unitary operator to commute with all p, it is required to be real, so the only remnant of the problems that plagued the transformations $\phi \to p\phi$ is that the time translation operator must be real, to provide conserved currents. This seems to not be an insurmountable problem.

Now consider the set of gauge transformations

$$
\{\phi(x) \to p(x)\phi(x)p(x)^*:\ p(x) \text{ a unit quaternion for all } x \in S\}
$$

as a set of unitarily implemented symmetries. Just as the global symmetry did not suffer the problems that plagued its counterpart, $\phi \to p\phi$, neither does the gauge theory. To show this, we note that any transformed field satisfies the spacelike-separated commutation relations because it is unitarily equivalent to a field that does. We will have restrictions, but they are not contradictions. They are that

$$\begin{aligned}[p(x)\phi(x)p(x)^*, p(y)\phi(y)p(y)^*]_\pm &= 0\\ [p(x)\phi(y)^+p(x)^*, p(y)\phi(y)p(y)^*]_\pm &= 0\end{aligned} \tag{3}$$

for all spacelike-separated x and y and for all functions p. We can find a field that satisfies these conditions. Take any world line W and then require that we can write $\phi(x)$ as $\phi_0(x)q(x)$ for $q(x)$ a quaternion and for $\phi_0(x)$ a real field operator satisfying the spacelike-separated commutation relations. If we further require that x not an element of W implies $q(x)=e_0$, then we have that $\phi(x)$ satisfies the restrictions (3) above. In fact, if $\phi_0(x)$ satisfies the CCR, then so do $\phi(x)$ and $p(x)\phi(x)p(x)$ for any function p. The only thing that seems to be amiss is that ϕ is not a continuous function of x, so that we cannot have unitary operators that represent and implement the Lorentz group continuously. This problem is, however, a product of the dependence of the field upon spacetime points rather than upon test functions (Streater and Wightman, 1964) and can be avoided by using a test function formulation.

So, the transformations $\phi \to p\phi p^*$ are the only transformations so far suggested as analogous to phase transformations that are consistent with what is basic to field theory—the commutation relations.

APPENDIX

If for some vector V, some unitary operator U, and some real a, $(U^nV, V)=a$ for $n=1, 2, 3, \ldots$, then $a\geq 0$.

Proof. It is sufficient to prove the theorem for normalized V. Define $V_1=UV-Va$ and $V_2=U^2V-Va$. Now

$$(V_1, V_2)=(UV, U^2V)-(UV, Va)-(Va, U^2V)+(Va, Va)=a-a^2$$

As Swartz's inequality holds for quaternionic Hilbert spaces (Horwitz and Bidenharn, 1984),

$$(a-a^2)^2=|(V_1, V_2)|^2\leq \|V_1\|^2\|V_2\|^2=(1-a^2)^2$$

Therefore, as $-1\leq a\leq 1$, $a^2\leq(1+a)^2$, and therefore $a\geq -1/2$. Writing $V'=V_1/\|V_1\|$ and $U'=U^2$, note that, for $n\in\{1,2,3,\ldots\}$,

$$\begin{aligned}(U'^nV', V') &= \frac{1}{\|V_1\|^2}(U^{2n+1}V-U^{2n}Va, UV-Va)\\ &= \frac{1}{\|V_1\|^2}(U^{2n+1}V, UV)-(U^{2n+1}V, Va)\end{aligned}$$

$$-(U^{2n}Va, UV)+(U^{2n}Va, Va)$$

$$=\frac{1}{1-a^2}a-2a^2+a^3$$

$$=a\frac{1-a}{1+a}$$

The same argument that applied to V, U, and a now applies to V', U', and $a'=a(1-a)/(1+a)$. So

$$a(1-a)/(1+a)\geq -1/2$$

Therefore, $3a-2a^2+1\geq 0$. Therefore, $a\geq -1/3$. In turn this means that

$$a'=a(1-a)/(1+a)\geq -1/3$$

If

$$a(1-a)/1+a\geq -1/m \qquad \text{for some} \quad m\in\{1,2,3,\ldots\}$$

then $a\geq -1/(m+1)$. So by induction it them follows that $a\geq -1/m$ for all $m\in\{1,2,3,\ldots\}$, and so $a\geq 0$.

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